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INTERPRETATION OF THE p-ADIC LOG GAMMA FUNCTION AND EULER CONSTANTS USING THE BERNOULLI MEASURE

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ABSTRACT. A regularized version of J. Diamond's p-adic log gamma function and his p-adic Euler constants are represented as integrals using B. Mazur's p-adic Bernoulli measure.

1. Introduction. Let $\mathbf{Z}_p$, $\mathbf{Z}_p^\times$, Ω_p denote, respectively, the p-adic integers, the p-adic integers not divisible by p, and the completion of the algebraic closure of the field of p-adic numbers. Let $|\ |_p$ be the absolute value on Ω_p with $|p|_p = p^{-1}$.

J. Diamond [3] defined a p-adic log gamma function

$$G_p(x) = \lim_{k\to\infty} p^{-k} \sum_{0\leq n<p^k} (x+n)\log_p(x+n) - (x+n) \quad \text{for } x \in \Omega_p - \mathbf{Z}_p$$

and a closely related function

$$G_p^*(x) = \lim_{k\to\infty} p^{-k} \sum_{0<n<p^k, p\nmid n} (x+n)\log_p(x+n) - (x+n) \qquad \text{for } x \in \Omega_p - \mathbf{Z}_p^\times$$

$$= G_p(x) - G_p(x/p) \quad \text{for } x \in \Omega_p - \mathbf{Z}_p.$$

Our purpose is to define a "regularized" version of G_p^* and show that it can be represented as a simple integral over $\mathbf{Z}_p^\times$ with respect to Mazur's p-adic Bernoulli measure μ_α, namely, as the "convolution" of μ_α with the p-adic logarithm (see (7) below). Recall (see [9], or Chapter II of [6]) that for $\alpha \in 1 + p\mathbf{Z}$, $\alpha \neq 1$, μ_α is defined by

$$\mu_\alpha(a + p^m\mathbf{Z}_p) = \alpha^{-1}[\alpha a p^{-m}] + (\alpha^{-1} - 1)/2.$$

(Actually, any $\alpha \in \mathbf{Z}_p^\times$ not a root of 1 can be used to regularize; but we shall take $\alpha \in 1 + p\mathbf{Z}$ for simplicity.) The moments of μ_α give the p-adic zeta-function ζ_p that was first defined by Kubota and Leopoldt [8].

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Our integral formula for the derivative of the regularized log gamma function (see (8) below) is an example of a very general phenomenon noticed by D. Barsky [2]: any Krasner analytic function on $\Omega_p - \mathbf{Z}_p^\times$ satisfying a certain growth condition is the "Cauchy transform" of some p-adic measure on $\mathbf{Z}_p^\times$. Theorem 2.1 below shows that, in the case of the regularized D-log gamma function, this measure is μ_α.

We shall also express Diamond's generalized p-adic Euler constants

$$\gamma_p(a, p^m) = -\lim_{k\to\infty} p^{-k} \sum_{0<n<p^k, n\equiv a (\mathrm{mod}\, p^m)} \log_p n \quad \text{if } p \nmid a,$$

$$\gamma_p = \gamma_p(0, 1) = \frac{p}{p-1} \sum_{a=1}^{p-1} \gamma_p(a, p) \tag{1}$$

as integrals.

2. Regularized log gamma function. Among the p-adic analogues of classical formulas which Diamond [3] derives for G_p is the "p-adic Stirling series"

$$G_p(x) = \left(x - \tfrac{1}{2}\right)\log_p x - x + \sum_{r=1}^{\infty} \frac{B_{r+1}}{r(r+1)x^r} \quad \text{for } |x|_p > 1, \tag{2}$$

where B_k is the kth Bernoulli number.

Let $l(x) = (x - \frac{1}{2})\log_p x - x$. Define operators T_p, T_α for $0 \neq \alpha \in \Omega_p$, by

$$T_p f(x) = f(x/p), \qquad T_\alpha f(x) = \alpha^{-1} f(\alpha x).$$

Then

$$(1 - T_p)G_p(x) = G_p^*(x) \quad \text{for } x \in \Omega_p - \mathbf{Z}_p, \tag{3}$$

$$(1 - T_\alpha)(1 - T_p)l(x) = -(1 - 1/p)x \log_p \alpha, \tag{4}$$

and, if we let $D = d/dx$,

$$DT_\alpha = \alpha T_\alpha D, \qquad DT_p = p^{-1}T_p D. \tag{5}$$

Let $A_r = \{x \in \Omega_p \mid |x - a|_p > r \text{ for all } a \in \mathbf{Z}_p^\times\}$. Thus, $A_1 = \{x \in \Omega_p \mid |x|_p > 1\}$. Choose $\alpha \in 1 + p\mathbf{Z}$, $\alpha \neq 1$. Define $G_{p,\alpha}$ on A_1 by

$$G_{p,\alpha} = (1 - T_\alpha)(1 - T_p)(G_p - l), \tag{6}$$

i.e., by (3) and (4),

$$G_{p,\alpha}(x) = (1 - T_\alpha)G_p^*(x) + (1 - 1/p)x \log_p \alpha \quad \text{for } x \in A_1.$$

THEOREM 2.1. *For* $x \in A_1$,

$$G_{p,\alpha}(x) = -\int_{\mathbf{Z}_p^\times} \log_p(x-t)\,\mu_\alpha(t); \tag{7}$$

$$D^r G_{p,\alpha}(x) = (-1)^r (r-1)! \int_{\mathbf{Z}_p^\times} \frac{\mu_\alpha(t)}{(x-t)^r} \quad \textit{for } r \geqslant 1$$

$$= (1-\alpha^r T_\alpha)(1-p^{-r}T_p) G_p^{(r)}(x) \quad \textit{for } r \geqslant 2. \tag{8}$$

PROOF. Using (6) and (2), we write the left side of (7) as

$$\sum_{r=1}^{\infty} \frac{1}{rx^r} (\alpha^{-r-1}-1)(1-p^r)\left(-\frac{B_{r+1}}{r+1}\right) = \sum_{r=1}^{\infty} \frac{1}{rx^r} \int_{\mathbf{Z}_p^\times} t^r \mu_\alpha(t),$$

by the fundamental property of μ_α, which allows it to be used to interpolate $\zeta(-r)$ [9]. Hence

$$G_{p,\alpha}(x) = \int_{\mathbf{Z}_p^\times} \sum_{r=1}^{\infty} \frac{(t/x)^r}{r} \mu_\alpha(t) = -\int_{\mathbf{Z}_p^\times} \log_p(1-t/x)\,\mu_\alpha(t)$$

$$= -\int_{\mathbf{Z}_p^\times} \log_p(x-t)\,\mu_\alpha(t),$$

since $\mu_\alpha(\mathbf{Z}_p^\times) = 0$. The formula for $D^r G_{p,\alpha}$ now follows immediately. Q.E.D.

COROLLARY 2.2. *For $x \in A_1$,*

$$G_p^{*\prime}(x) - G_p^{*\prime}(\alpha x) = -(1-1/p)\log_p \alpha - \int_{\mathbf{Z}_p^\times} \frac{\mu_\alpha(t)}{x-t}$$

$$= -\int_{\mathbf{Z}_p^\times} \left(\frac{1}{t} + \frac{1}{x-t}\right)\mu_\alpha(t).$$

The first equality in the corollary follows from (4) and (8), and the second follows from formula (12) in §4 below.

We now use (7) to *define* $G_{p,\alpha}$ for $x \in \Omega_p - \mathbf{Z}_p^\times$. This integral exists for all such x, since with $x \notin \mathbf{Z}_p^\times$ fixed, $\log_p(x-t)$ is continuous in $t \in \mathbf{Z}_p^\times$.

COROLLARY 2.3. *With $G_{p,\alpha}$ defined by* (7),

$$G_{p,\alpha}(0) = (1-\alpha^{-1})L_p'(0,\omega),$$

where L_p is the p-adic L-function [5] *and ω is the Teichmüller character.*

In fact, the right-hand side equals

$$\lim_{k\to\infty} \frac{1-\alpha^{-1}}{-(p-1)p^k} L_p(-(p-1)p^k, \omega)$$

$$= -\lim_{k\to\infty} \frac{1-\alpha^{-1}}{(p-1)p^k} \frac{1}{1-\alpha^{-(p-1)p^k-1}}$$

$$\cdot \int_{\mathbf{Z}_p^\times} \exp\{(p-1)p^k \log_p t\}\,\mu_\alpha(t)$$

$$= -\int_{\mathbf{Z}_p^\times} \log_p t\,\mu_\alpha(t) = G_{p,\alpha}(0).$$

3. Analytic continuation. Diamond [3] proved that $G_p'' = D^2G_p$ is analytic in the sense of Krasner [7] on $\Omega_p - Z_p$, but he noted that G_p and G_p' are not. However, for our regularized $G_{p,\alpha}$ already the first derivative is Krasner analytic.

THEOREM 3.1. *$DG_{p,\alpha}$ is Krasner analytic on $\Omega_p - \mathbf{Z}_p^\times$.*

PROOF. Since $\Omega_p - \mathbf{Z}_p^\times = \bigcup_{m=0}^{\infty} A_{p^{-m}}$, it suffices to show that for m fixed, $f(x) = -DG_{p,\alpha}(x)$ is a uniform limit of rational functions on $A_{p^{-m}}$ without poles there. Since

$$f = \sum_{0<a<p^{m+1}, p\nmid a} f_a, \quad \text{where } f_a(x) = \int_{a+p^{m+1}\mathbf{Z}_p} \frac{\mu_\alpha(t)}{x-t},$$

it suffices to show this for f_a. For $t = a + p^{m+1}s$, $s \in \mathbf{Z}_p$, we have

$$\frac{1}{x-t} = \frac{1}{x-a} \sum_{j=0}^{\infty} \left(\frac{p^m}{x-a}\right)^j p^j s^j,$$

so that

$$f_a(x) = \frac{1}{x-a} \sum_{j=0}^{\infty} \left(\frac{p^m}{x-a}\right)^j p^j \int_{\mathbf{Z}_p} s^j \mu_\alpha(a + p^{m+1}s).$$

Since $|p^m/(x-a)|_p < 1$ on $A_{p^{-m}}$ and $|\int_{\mathbf{Z}_p}|_p \leqslant 1$, it follows that f_a is in fact a uniform limit of rational functions on $A_{p^{-m}}$. Q.E.D.

COROLLARY 3.2. *For all $x \in \Omega_p - \mathbf{Z}_p^\times$ and all $r \geqslant 2$,*

$$(1 - \alpha^r T_\alpha) G_p^{*(r)}(x) = (-1)^r (r-1)! \int_{\mathbf{Z}_p^\times} \frac{\mu_\alpha(t)}{(x-t)^r}.$$

In fact, Diamond's argument proving Krasner analyticity of G_p'' on $\Omega_p - \mathbf{Z}_p$ will also prove Krasner analyticity of $G_p^{*\prime\prime}$ and all higher derivatives on $\Omega_p - \mathbf{Z}_p^\times$. Since both sides of the equality are Krasner analytic on $\Omega_p - \mathbf{Z}_p^\times$ and agree on A_1, they must be equal on all of $\Omega_p - \mathbf{Z}_p^\times$.

COROLLARY 3.3 (SEE [4]). *$D^r G_p^*(0) = -(r-1)! L_p(r, \omega^{1-r})$ for $r \geqslant 2$.*

In fact, by Corollary 3.2, the left side equals

$$-(\alpha^{r-1} - 1)^{-1} (r-1)! \int_{\mathbf{Z}_p^\times} t^{-r} \mu_\alpha(t) = -(r-1)! L_p(r, \omega^{1-r}).$$

Questions. 1. In [4, Propositions 4 and 5], Diamond proved the following relation for $L_p(r, \chi\omega^{1-r})$ when χ is a Dirichlet character mod p^m, $m \geqslant 1$, $r \geqslant 2$:

$$L_p(r, \chi\omega^{1-r}) = \frac{(-1)^r}{p^{mr}(r-1)!} \sum_{0<a<p^m, p\nmid a} \chi(a) D^r G_p(a/p^m). \tag{9}$$

Can this expression be derived using the integral formulas? The difficulty comes when trying to express $D^r G_p(a/p^m)$ in terms of $D^r G_{p,\alpha}$ when $a \neq 0$.

2. Does Corollary 3.2 hold when $r = 0, 1$, i.e., do we have

$$G_p^*(x) - \alpha^{-1} G_p^*(\alpha x) = -(1 - 1/p) x \log_p \alpha - \int_{\mathbf{Z}_p^\times} \log_p(x - t)\, \mu_\alpha(t),$$

$$G_p^{*\prime}(x) - G_p^{*\prime}(\alpha x) = -\int_{\mathbf{Z}_p^\times} \left(\frac{1}{t} + \frac{1}{x - t} \right) \mu_\alpha(t) \tag{10}$$

for *all* $x \in \Omega_p - \mathbf{Z}_p^\times$? Is $G_p^{*\prime}(x) - G_p^{*\prime}(\alpha x)$ Krasner analytic on $\Omega_p - \mathbf{Z}_p^\times$? In particular, can Corollary 2.3 be rewritten simply: $G_p^*(0) = L_p'(0, \omega)$? Note that when $x \in p\mathbf{Z}$, the left side of (10) can be rewritten

$$\lim_{k \to \infty} p^{-k} \sum_{0 < n < p^k, p \nmid n} \log_p(n + x) - \log_p(n + \alpha x) = - \sum_{|x| < n < |\alpha x|, p \nmid n} \frac{1}{n}$$

(here $|\ |$ means ordinary archimedean absolute value of an integer).

We obtain a partial affirmative answer to the second question in the following

THEOREM 3.4. *$G_p^{*\prime}(x) - G_p^{*\prime}(\alpha x)$ is Krasner analytic on $A_{|\alpha - 1|_p}$, and for $x \in A_{|\alpha - 1|_p}$,*

$$G_p^{*\prime}(x) - G_p^{*\prime}(\alpha x) = -(1 - 1/p) \log_p \alpha - \int_{\mathbf{Z}_p^\times} \frac{\mu_\alpha(t)}{x - t}$$

$$= -\int_{\mathbf{Z}_p^\times} \left(\frac{1}{t} + \frac{1}{x - t} \right) \mu_\alpha(t).$$

PROOF. Since $A_{|\alpha - 1|_p} = \bigcup_{r > |\alpha - 1|_p} A_r$, it suffices to write $f(x) = G_p^{*\prime}(x) - G_p^{*\prime}(\alpha x)$ as a uniform limit of rational functions on A_r for fixed $r > |\alpha - 1|_p$. We have

$$f(x) = \lim_{k \to \infty} p^{-k} \sum_{0 < n < p^k, p \nmid n} \log_p(x + n) - \log_p(\alpha x + n)$$

$$= -(1 - 1/p) \log_p \alpha - \lim_{k \to \infty} p^{-k} \sum_{0 < n < p^k, p \nmid n} \log_p \frac{x + n/\alpha}{x + n}.$$

Thus, if we let $\alpha' = 1 - 1/\alpha$ and $f_n(x) = \log_p(1 - \alpha' n/(x + n))$, we have

$$f(x) = -(1 - 1/p) \log_p \alpha - \lim_{k \to \infty} p^{-k} \sum_{0 < n < p^k, p \nmid n} f_n(x),$$

where the limit is uniform on A_r. But, since $|\alpha' n/(x + n)|_p < |\alpha - 1|_p / r < 1$ for $x \in A_r$, it follows that each

$$f_n(x) = -\sum_{j=1}^{\infty} \frac{1}{j} \left(\frac{\alpha' n}{x + n} \right)^j$$

is a uniform limit of rational functions. Q.E.D.

COROLLARY 3.5. *For* $|x|_p < 1$,

$$G_p^{*\prime}(x) - G_p^{*\prime}(\alpha x) = \int_{\mathbf{Z}_p^\times} \sum_{j=1}^{\infty} \frac{x^j}{t^{j+1}} \mu_\alpha(t)$$

$$= \sum_{j=1}^{\infty} (1-\alpha^j)(1-p^{-j-1}) L_p(j+1, \omega^{-j}) x^j.$$

4. Euler constants. In [3] Diamond defined generalized p-adic Euler constants by (1) above and proved that

$$\frac{1}{p^{m-1}(p-1)} \sum_{\chi \neq \chi_0} \bar{\chi}(a) L_p(1, \chi) = \gamma_p(a, p^m) - p^{-m}\gamma_p \quad \text{for } p \nmid a. \tag{11}$$

Once we express γ_p in terms of the p-adic zeta-function ζ_p–actually, it will equal the Euler constant one would expect from a zeta-function–we can express γ_p and, hence, $\gamma_p(a, p^m)$ as integrals.

Let

$$\tilde{\gamma}_p \underset{\text{def}}{=} \frac{p}{p-1} \lim_{\varepsilon \to 0} \left(\zeta_p(1+\varepsilon) - (1-1/p)/\varepsilon\right)$$

$$= \frac{p}{p-1} \lim_{N \to \infty} \left(\zeta_p(1-(p-1)p^N) + (1-1/p)/(p-1)p^N\right)$$

$$= \frac{p}{p-1} \lim_{N \to \infty} \left[(1-p^{(p-1)p^N-1}) \left(-\frac{B_{(p-1)p^N}}{(p-1)p^N} \right) + p^{-N-1} \right]$$

$$= -\frac{p}{p-1} \lim_{N \to \infty} \left(\frac{B_{(p-1)p^N}}{(p-1)p^N} - p^{-N-1} \right).$$

We claim that $\tilde{\gamma}_p = \gamma_p$. In fact, Kubota and Leopoldt [**8**, §3] prove that if $A(u) = \sum_{n=0}^{\infty} a_n (u-1)^n$ converges for $|u-1|_p < 1/p$ ($< \frac{1}{4}$ if $p = 2$), and if we let

$$M^k(A) \underset{\text{def}}{=} p^{-k} \sum_{\substack{0 < i < p^k \\ p \nmid i}} A(i/\omega(i)),$$

$$M(A) = \lim_{k \to \infty} M^k(A), \quad L(u) = \sum_{n=1}^{\infty} (-1)^{n-1}(u-1)^n/n,$$

then

$$\zeta_p(s) = \frac{1}{s-1} \sum_{n=0}^{\infty} \frac{M(L^n)}{n!} (1-s)^n$$

$$= \frac{1}{s-1}\left(1 - \frac{1}{p}\right) + \frac{1}{s-1} \sum_{n=1}^{\infty} \frac{M(L^n)}{n!} (1-s)^n.$$

Using this, we have

$$\left(1-\frac{1}{p}\right)\tilde{\gamma}_p = \lim_{\varepsilon\to 0}\left(\zeta_p(1+\varepsilon)-\left(1-\frac{1}{p}\right)/\varepsilon\right) = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\sum_{n=1}^{\infty}\frac{M(L^n)}{n!}(-\varepsilon)^r$$

$$= -M(L) = -\lim_{k\to\infty} p^{-k}\sum_{\substack{0<i<p^k\\ p\nmid i}}\log_p i = \left(1-\frac{1}{p}\right)\gamma_p.$$

This proves the claim. (The above proof, which is shorter and neater than my original proof, was kindly given me by the referee.)

Thus, Diamond's p-adic Euler constant agrees with the one from the Kubota-Leopoldt zeta-function.

We now derive integral formulas for γ_p and $\gamma_p(a, p^m)$. In what follows we now suppose $\alpha \in 1 + p^m\mathbf{Z}$, $\alpha \neq 1$, and χ is a Dirichlet character mod p^m. For small s we use

$$L_p(1+s,\chi) = (\alpha^s-1)^{-1}\int_{\mathbf{Z}_p^\times} t^{-s-1}\omega^s(t)\chi(t)\,\mu_\alpha(t)$$

$$= (\alpha^s-1)^{-1}\int_{\mathbf{Z}_p^\times}\exp\{-s\log_p t\}\frac{\chi(t)}{t}\,\mu_\alpha(t).$$

Note that $(\alpha^s-1)/s \to \log_p\alpha$ as $s\to 0$. Let $\varepsilon(\chi) = 1$ if $\chi = \chi_0$, 0 otherwise. Then

$$\frac{1}{\log_p\alpha}\int_{\mathbf{Z}_p^\times}\frac{\chi(t)}{t}\,\mu_\alpha(t) = \lim_{s\to 0} sL_p(1+s,\chi) = (1-1/p)\varepsilon(\chi). \tag{12}$$

Define $\gamma_p(\chi)$ by

$$(1-1/p)\gamma_p(\chi) = \begin{cases}\lim\limits_{s\to 0}\left(L_p(1+s,\chi)-\dfrac{1-1/p}{s}\,\varepsilon(\chi)\right)\\ (1-1/p)\gamma_p \quad \text{if } \chi=\chi_0,\\ L_p(1,\chi) \quad \text{otherwise.}\end{cases}$$

Then

$$(1-1/p)\gamma_p(\chi) = \lim_{s\to 0}\int_{\mathbf{Z}_p^\times}\frac{1}{\alpha^s-1}\,\frac{e^{-s\log_p t}\chi(t)}{t} - \frac{\chi(t)}{st\log_p\alpha}\,\mu_\alpha(t)$$

$$= \lim_{s\to 0}\int_{\mathbf{Z}_p^\times}\left(\frac{e^{-s\log_p t}}{e^{s\log_p\alpha}-1}-\frac{1}{s\log_p\alpha}\right)\frac{\chi(t)}{t}\,\mu_\alpha(t)$$

$$= \lim_{s\to 0}\frac{1}{s\log_p\alpha}\int_{\mathbf{Z}_p^\times}\left(\frac{1-s\log_p t}{1+(s/2)\log_p\alpha}-1\right)\frac{\chi(t)}{t}\,\mu_\alpha(t)$$

$$= -\int_{\mathbf{Z}_p^\times}\left(\frac{\log_p t}{\log_p\alpha}+\frac{1}{2}\right)\frac{\chi(t)}{t}\,\mu_\alpha(t).$$

Since (11) gives us

$$(1 - 1/p)\gamma_p(a, p^m) = p^{-m}\left((1 - 1/p)\gamma_p + \sum_{\chi \neq \chi_0} \bar{\chi}(a) L_p(1, \chi)\right)$$

$$= p^{-m}(1 - 1/p) \sum_{\text{all } \chi} \bar{\chi}(a)\gamma_p(\chi)$$

$$= -p^{-m} \int_{\mathbf{Z}_p^\times} \left(\frac{\log_p t}{\log_p \alpha} + \frac{1}{2} \right) \frac{1}{t} \sum_{\chi} \chi(t/a)\, \mu_\alpha(t),$$

we may conclude

THEOREM 4.1.

$$(1 - 1/p)\gamma_p(a, p^m) = -\int_{a + p^m \mathbf{Z}_p} \left(\frac{\log_p t}{\log_p \alpha} + \frac{1}{2} \right) \frac{\mu_\alpha(t)}{t}.$$

REMARK. Note that

$$G_p'(a/p^m) = \lim_{k \to \infty} p^{-k} \sum_{0 < n < p^k} \log_p\left(\frac{a}{p^m} + n \right) = -p^m \gamma_p(a, p^m),$$

and, similarly,

$$G_p^{*\prime}(0) = -(1 - 1/p)\gamma_p.$$

Using Diamond's relation (9) in the same way as we used (11) to prove Theorem 4.1, we see that for $r \geqslant 2$,

$$G_p^{(r)}(a/p^m) = p^{mr}(r-1)!\,(-1)^r(\alpha^{r-1} - 1)^{-1} \int_{a + p^m \mathbf{Z}_p} \frac{\mu_\alpha(t)}{t^r}.$$

Namely, replace the left-hand side of (9) by $(\alpha^{r-1} - 1)^{-1} \int_{\mathbf{Z}_p^\times} \chi(t) t^{-r} \mu_\alpha(t)$. Then let χ run over all characters mod p^m, for each χ multiply (9) by $\bar{\chi}(a)$, and take the sum. One obtains

$$(p^m - p^{m-1})(\alpha^{r-1} - 1)^{-1} \int_{a + p^m \mathbf{Z}_p} t^{-r} \mu_\alpha(t)$$

$$= \frac{(-1)^r}{p^{mr}(r-1)!} (p^m - p^{m-1}) D^r G_p(a/p^m).$$

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